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A new functional perturbation method for linear non-homogeneous materials

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Abstract

A functional perturbation method (FPM), for solving boundary value problems of linear materials with non-homogeneous properties is introduced. The FPM is based on considering the unknown field such as displacements or temperatures, as a functional of the non-uniform property, i.e., elastic modulus or thermal conductivity. The governing differential equations are expanded functionally by Fréchet series, leading to a set of differential equations with constant coefficients, from which the unknown field is found successively to any desirable degree of accuracy. A unique property of the FPM is that once the Fréchet functions are found, the solution for any morphology is obtained by direct integration, without re-solving the differential equation for each case. The FPM procedure is outlined first for general linear differential equations with non-uniform coefficients. Then, four examples are solved and discussed: a 1D tensile loading of a rod with continuously varying and discontinuous moduli, beam bending, beam deflection on non-uniform elastic foundation and a unidirectional heat conduction problem. FPM results are compared with the exact (if exists) or numerical solution. The FPM accuracy for the bending problem is also compared to the common Rayleigh–Ritz and Galerkin methods. It is shown that the FPM is inherently more accurate, since the convergence rate of the other methods depends on the arbitrarily chosen shape functions, while in the FPM, these functions are obtained as generic results of each order of the solution. The FPM solution is analytical, and is shown to be suitable for large variations in material properties. Thus, a direct insight of each functional perturbation order is possible. Advantages and limitations of the FPM as compared to other existing methods are discussed in detail.

Keywords: Functional perturbation method; Heterogeneity; Composite materials; Linear differential equations; Non-uniform coefficients

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1. Introduction and motivation

In many practical applications, it is desired to design, analyze or optimize a non-homogeneous structure, i.e., its material properties vary in space. One of the latest examples is the "functionally graded materials" for improved design. Another "old" example is when the heterogeneity is driven by varying dimensions, such as for buckling or vibrations of beams and columns with non-uniform cross sections (Elishakoff, 2000). Mathematically, the problem is of solving a set of governing equations with non-uniform coefficients. These types of problems, even when the differential equations are linear, have exact solutions only for special classes. In addition to the common numerical (FE) or energy (Rayligh–Ritz) methods, more advanced analytical or semi-analytical approaches have been developed to solve such problems: self-conjugate method (Li, 2001), Semi-inverse method (Elishakoff and Candan, 2001; Guede and Elishakoff, 2001), exact finite elements (Dugush and Eisenberger, 2002), special shape functions (Au et al., 1999; Esmailzadeh and Ohadi, 2000), limit analysis (Lee and Hsiao, 2002), successive approximation (Elishakoff et al., 2002), combined numerical–analytical method (Chen, 2003) and more. Homogenization method is also used for periodic heterogeneity (Buannic and Cartraud, 2001). However, all of the above are either limited to a certain class of non-uniformity, or suffer from accuracy problems (convergence) due to the fact that displacement shape functions are arbitrarily chosen. These limitations are especially important when local heterogeneities (grain-like materials) and irregularities are involved.

In this study, a new method for solving a single linear equation (ODE or PDE) or a set of linear equation with nonuniform coefficient is proposed, based on *functional* perturbations. Perturbation methods for inhomogeneous materials have been developed by transforming the differential equations to an integral form and then using projection operators (see review by Fokin, 1996). The proposed functional perturbation method (FPM), on the other hand, uses the original differential equation directly, leading to a set of successive equations for each order of accuracy. The FPM has been initially developed for elastic problems of stochastically heterogeneous beams (Altus, 2001, 2003; Altus and Givli, 2003; Altus and Totry, 2003). However, in these studies the original differential equations were replaced by simplified algebraic equations, after applying some energy based approximations. Therefore, only limited "macro" characteristics (such as reactions, or buckling forces) could be estimated accurately.

The FPM is generalized herein, by operating *directly* on the differential equations, so that all micro-details of the structural behavior can be found. Moreover, the solution is analytical, permitting a better insight into general "micro-effects".

Some mathematical definitions and notations are introduced herein by way of introduction. A function (say, u) of x and its regular derivatives will be written as

$$u(x) \equiv u_x, \quad \frac{\mathrm{d}u(x)}{\mathrm{d}x} \equiv u_{x,x}, \quad \frac{\mathrm{d}^n u(x)}{\mathrm{d}x^n} \equiv u_{x,x^n}. \tag{1.1}$$

If u is also functional of E, which is a function of x_m

$$\frac{\delta u(x, \{E(x_m)\})}{\delta E(x_n)} \equiv u_{x,E_n} \tag{1.2}$$

{ } is for functional relation and the derivative (1.2) is a function of both x and x_n. Multiple functional differentiations are denoted as:

$$\frac{\delta^2 u(x, \{E(x_m)\})}{\delta E(x_1)\delta E(x_2)} \equiv u_{x,E_1E_2}, \text{ etc.} \tag{1.3}$$

The Dirac operator will be used frequently in the text. Its differential definition is especially convenient (Beran, 1968):

$$\delta_{xx_1} = \frac{\delta E_x}{\delta E_{x_1}}. \tag{1.4}$$

The use of δ for two different purposes as in (1.4) will not cause any ambiguity. The derivative of the Dirac operator is denoted similarly

$$\frac{\delta E_{x,x}}{\delta E_{x_1}} = \left(\frac{\delta E_x}{\delta E_{x_1}}\right)_{,x} = \delta_{xx_1,x}. \tag{1.5}$$

Notice the symmetry relations

$$\delta_{xx_1} = \delta_{x_1x} \rightarrow \delta_{xx_1,x} = -\delta_{xx_1,x_1}, \text{ etc.} \tag{1.6}$$

More on Generalized Functions can be found elsewhere (Kanwal, 1998). Integrations are denoted by the convolution sign. For example

$$\int_{x=0}^{1} v_x \,\mathrm{d}x \equiv v_x \overset{1}{\underset{x=0}{*}} 1_x, \tag{1.7}$$

where 1 is a unit function. Inner products of vectors are denoted similarly

$$\sum_{k=1}^{n} u_k \equiv u_k \overset{n}{\underset{1}{:}} 1_k, \tag{1.8}$$

etc. 1_k are vector (or series) elements of unit size.

2. Theoretical considerations

Consider a linear differential equation with non-uniform coefficients of the form:

$$L = (\phi, u) = \phi_{(0)}(E)\cdot u + \phi_{(1)}(E)\cdot u_{,x} + \phi_{(2)}(E)\cdot u_{,xx} + \cdots = f(x). \tag{2.1a}$$

For convenience, write

$$L = \phi_{(k)} \cdot u_{,x^k}, \tag{2.1b}$$

where the common index summation convention is applied. $E = E(x)$ is a given function, usually some property of a non-homogeneous material and $u(x)$ is the unknown function (cf. displacements in elastic problems). $\phi_{(i)}$ is a given set which may include nonlinear functions, functionals or derivatives of $E(x)$ and f is the "loading". Boundary conditions are

$$g_1(u)|_{x=0} = a, \quad g_2(u)|_{x=1} = b. \tag{2.2}$$

Generalizing the functional perturbation method (Altus, 2001), u is considered to be a *functional* of morphology E, since any change in $E(x_1)$ can affect $u(x_2)$

$$u = u(\{E(x_m)\}, x). \tag{2.3}$$

By definition, u satisfies (2.1) for *any* given E. Define a convenient uniform (average in some sense) function $\langle E\rangle$, say,

$$\langle E\rangle = \int_0^1 E(x)\,\mathrm{d}x = E * 1. \tag{2.4}$$

Then, denote the perturbation (deviation) function $E'(x)$ as

$$E'(x) = E(x) - \langle E\rangle. \tag{2.5}$$

Denote also

$$u^{(0)} = u(\{\langle E\rangle\}, x), \quad \phi^{(0)} = \phi(\langle E\rangle), \quad \phi^{(0)} \equiv \phi^{(0)}_{(0)}, \phi^{(0)}_{(1)}, \phi^{(0)}_{(2)}, \text{ etc.} \tag{2.6}$$

For the special homogeneous case $E = \langle E\rangle$, (2.1) reduces to

$$(\phi^{(0)}, u^{(0)}) = \phi^{(0)}_{(i)} \cdot u^{(0)}_{,x^i} = f(x). \tag{2.7}$$

Thus, $u^{(0)}(x)$ is the solution of (2.1) under the boundary conditions (2.2).

The FPM is essentially a generalization of the regular (parametric) perturbation methods (Hinch, 1994). The unknown function u, as well as ϕ, is expanded as a Fréchet series around $\langle E\rangle$:

$$u = u(\langle E\rangle, x) + u_{,E_1} * E'_1 + \frac{1}{2} u_{,E_1E_2} * * E'_1E'_2 + \cdots \tag{2.8}$$

$$\phi = \phi(\langle E\rangle, x) + \phi_{,E_1} * E'_1 + \frac{1}{2} \phi_{,E_1E_2} * * E'_1E'_2 + \cdots \tag{2.9}$$

All functional derivatives are at $E = \langle E\rangle$and (*) is the common convolution symbol. Information about E is given, so we have to calculate the functional derivatives of u at $E = \langle E\rangle$. To achieve this goal, expand L functionally:

$$L = L(\langle E\rangle, u(\langle E\rangle x)) + L_{,E_1} * E'_1 + \frac{1}{2} L_{,E_1E_2} * * E'_1E'_2 + \cdots = f(x) \tag{2.10}$$

Since (2.10) must hold for *any* $E'(x)$ and $f(x)$ is given

$$L_{,E_1}|_{\langle E\rangle} = 0, \quad L_{,E_1E_2}|_{\langle E\rangle} = 0, \cdots \tag{2.11}$$

The above (2.11) series of differential equations are solved with homogeneous boundary conditions. From (2.1b)

$$L_{,E_1} = \phi_{(i),E_1} u_{,x^i} + \phi_{(i)} \cdot u_{,x^iE_1}. \tag{2.12}$$

From (2.8)

$$u_{,x^i}|_{E=\langle E\rangle} = (u^{(0)})_{,x^i} \equiv u^{(0)}_{,x^i}. \tag{2.13}$$

For convenience, also denote

$$u_{,E_1}|_{\langle E\rangle} * E'_1 = u^{(1)}, \quad \phi_{,E_1}|_{\langle E\rangle} * E'_1 = \phi^{(1)} \text{ etc.} \tag{2.14}$$

Applying (2.11) and using (2.12)–(2.14)

$$L^{(1)} = L_{,E_1}|_{\langle E\rangle} * E'_1 = \phi^{(1)}_{(i)} \cdot u^{(0)}_{,x^i} + \phi^{(0)}_{(i)} \cdot u^{(1)}_{,x^i} = 0. \tag{2.15}$$

Since $u^{(0)}$ is known from the solution of (2.7) and (2.15) is a new differential equation for $u^{(1)}$

$$(\phi^{(0)}, u^{(1)}) = \phi^{(0)}_{(i)} \cdot u^{(1)}_{,x^i} = f^{(1)}, \quad f^{(1)} = -\phi^{(1)}_{(i)} \cdot u^{(0)}_{,x^i}. \tag{2.16}$$

It is seen that (2.16) differs from (2.7) by the RHS only. Since the boundary conditions are already fulfilled by $u^{(0)}$, we find $u^{(1)}$ by solving (2.16) with homogeneous boundary conditions.

Knowing $u^{(0)}$ and $u^{(1)}$, we solve for $u^{(2)}$ by the second-order equation (2.11b)

$$L_{,E_1E_2} = \left(\phi_{(i),E_1E_2} u_{,x^i} + \phi_{(i),E_2} \cdot u_{,x^iE_1}\right) + \left(\phi_{(i),E_1} u_{,x^iE_2} + \phi_{(i)} \cdot u_{,x^iE_1E_2}\right) = 0. \tag{2.17}$$

Convolution twice by $E(x_1)$ and $E(x_2)$ around $E = \langle E \rangle$

$$L_{,E_1E_2} * *E'_1E'_2 = \left(\phi_{(i),E_1E_2}u_{,x^i} * *E'_1E'_2 + \phi_{(i),E_2} * E'_2 \cdot u_{,x^iE_1} * E'_1\right) + \left(\phi_{(i),E_1} * E'_1u_{,x^iE_2} * E'_2 + \phi_{(i)} \cdot u_{,x^iE_1E_2} * *E'_1E'_2\right) = 0. \tag{2.18}$$

Note that

$$\phi_{(i),E_1} * E'_1 = \phi_{(i),E_2} * E'_2 \equiv \phi^{(1)}_{(i)}, \quad u_{,x^iE_1} * E'_1 = u_{,x^iE_2} * E'_2 \equiv u^{(1)}_{,x^i}. \tag{2.19}$$

Similarly, denote

$$\phi^{(2)}_{(i)} \equiv \phi_{(i),E_1E_2} * *E'_1E'_2, \quad u^{(2)}_{,x^i} \equiv u_{,x^iE_1E_2} * *E'_1E'_2. \tag{2.20}$$

Then, (2.18) reduces to the form:

$$\phi^{(2)}_{(i)} \cdot u^{(0)}_{,x^i} + 2 \cdot \phi^{(1)}_{(i)} \cdot u^{(1)}_{,x^i} + \phi^{(0)}_{(i)} \cdot u^{(2)}_{,x^i} = 0. \tag{2.21}$$

Since $u^{(0)}$ and $u^{(1)}$ are known from the solution of (2.7) and (2.16), we are left again with a differential equation for $u^{(2)}$

$$\phi^{(0)}_{(i)} \cdot u^{(2)}_{,x^i} = f^{(2)}, \quad f^{(2)} = -(\phi^{(2)}_{(i)} \cdot u^{(0)}_{,x^i} + 2 \cdot \phi^{(1)}_{(i)} \cdot u^{(1)}_{,x^i}). \tag{2.22}$$

Examining the form of (2.7), (2.15) and (2.21), it is straightforward to show that the third-order equation is

$$\phi^{(3)}_{(i)} \cdot u^{(0)}_{,x^i} + 3 \cdot \phi^{(2)}_{(i)} \cdot u^{(1)}_{,x^i} + 3 \cdot \phi^{(1)}_{(i)} \cdot u^{(2)}_{,x^i} + \phi^{(0)}_{(i)} \cdot u^{(3)}_{,x^i} = 0 \tag{2.23}$$

Finally, for the n'th-order

$$\sum_{k=0}^{n} C^n_k \cdot \phi^{(n-k)}_{(i)} \cdot u^{(k)}_{,x^i} = 0, \quad C^n_k = \frac{n!}{k!(n-k)!}. \tag{2.24}$$

C^n_k is the permutation number. The solution is then

$$u_x = \sum_{k=0}^{n} \frac{1}{k!} u^{(k)}_x \equiv u^{(k)} \cdot 1^{(k)}. \tag{2.25}$$

It is important to emphasize, that unlike the existing approximation methods (Galerkin, Rayleigh–Ritz, etc.), where the accuracy of the approximate solution depends on the choice of the shape functions, the FPM "produces" shape functions which are "generated" inherently for each problem; this is a *fundamental* difference. Another important feature is that since the solution of $u^{(i)}$ is obtained analytically, we can explicitly find the functional derivatives $u_{,E'_1}, u_{,E'_1E'_2}$, etc. Thus, the solution for *any* morphology E'_1, $E'_1E'_2$, etc. can be found by convoluting the two series, *without* re-solving the problem for each case.

3. Case study 1: a 1D elastic rod with variable stiffness

3.1. General considerations

Consider an elementary 1D case of a heterogeneous rod under unidirectional tension. The governing differential equation is

$$L = -\frac{\mathrm{d}}{\mathrm{d}x}\left(E(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right) = 0 \rightarrow E_{,x}u_{,x} + Eu_{,xx} = 0. \tag{3.1}$$

Boundary conditions are "mixed"

$$g_1(u) = u|_{x=0} = 0, \quad g_2(u) = \sigma_{xx}(u) = E(x)\frac{\mathrm{d}u}{\mathrm{d}x}|_{x=1} = p. \tag{3.2}$$

In this case

$$\phi_{(0)} = 0, \quad \phi_{(1)} = E_{,x}, \quad \phi_{(2)} = E, \quad f(x) = 0. \tag{3.3}$$

By (2.6)

$$\phi^{(0)}_{(0)} = 0, \quad \phi^{(0)}_{(1)} = 0, \quad \phi^{(0)}_{(2)} = \langle E\rangle. \tag{3.4}$$

Following (2.7), the zero-order equation is

$$\langle E\rangle \cdot u^{(0)}_{,x^2} = \langle E\rangle \cdot u^{(0)}_{,xx} = 0. \tag{3.5}$$

Using (2.2) and (3.2), the zero-order solution is

$$u^{(0)}_x = \frac{p}{E_{x=1}}x. \tag{3.6}$$

To solve the first-order equation we need

$$\phi^{(1)}_{(0)} = \phi_{(0),E_1}|_{\langle E\rangle} * E'_1 = 0, \tag{3.7}$$

$$\phi^{(1)}_{(1)} = \phi_{(1),E_1}|_{\langle E\rangle} * E'_1 = E_{,xE_1}|_{\langle E\rangle} * E'_1 = \delta_{xx_1,x} * E'_1 = E'_{x,x}, \tag{3.8}$$

$$\phi^{(1)}_{(2)} = \phi_{(2),E_1}|_{\langle E\rangle} * E'_1 = E_{,E_1}|_{\langle E\rangle} * E'_1 = \delta_{xx_1} * E'_1 = E'_x. \tag{3.9}$$

Inserting (3.7)–(3.9) in (2.16)

$$\phi^{(0)}_{(i)} \cdot u^{(1)}_{,x^i} = \langle E\rangle \cdot u^{(1)}_{,xx} = f^{(1)}, \quad f^{(1)} = -E'_{x,x} \cdot u^{(0)}_{,x} - E'_x u^{(0)}_{,xx} = -\frac{p}{E_{x=1}} \cdot E'_{x,x}. \tag{3.10}$$

The final equation for the first-order is

$$u^{(1)}_{,xx} = -\frac{p}{\langle E\rangle E_{x=1}} \cdot E'_{x,x} \rightarrow u^{(1)} = -\frac{p}{\langle E\rangle E_{x=1}} \cdot \left[\left(E'_\xi \overset{x}{\underset{\xi=0}{*}} 1\right) - E'_{x=1}\right]. \tag{3.11}$$

Boundary conditions for the first- and higher-order equations are

$$u^{(1)}_{x=0} = 0, \quad u^{(1)}_{x,x}|_{x=1} = 0. \tag{3.12}$$

Following the above procedure we obtain a recursive differential equation for the kth-order

$$\langle E\rangle \cdot u^{(k)}_{,xx} = f^{(k)}, \quad f^{(k)} = -E'_{x,x} \cdot u^{(k-1)}_{,x} - E'_x u^{(k-1)}_{,xx} \tag{3.13}$$

with the homogeneous boundary conditions

$$u^{(k)}_{x=0} = 0, \quad u^{(k)}_{x,x}|_{x=1} = 0. \tag{3.14}$$

The reason for this simple recursive formula, in which each step depends only on the previous one and all steps involve the same differential equation, comes from the linearity of ϕ as a function of E and its derivatives (with respect to x), i.e.,

$$\phi_{(m)} = A_{(i)}E_{,x^i} \rightarrow \phi_{(m),E_1E_2\ldots} = 0. \quad (3.15)$$

Moreover, the invariance of the differential equation permits using a single Green function for the solution of all orders (k) of (3.13) and (3.14). The Green function is

$$G(\xi, x) = \frac{1}{\langle E \rangle}\begin{cases} -\xi, & 0 < \xi < x, \\ -x, & x < \xi < 1. \end{cases} \quad (3.16)$$

G is the solution of

$$\langle E \rangle \cdot G_{\xi x,\xi\xi} = \delta_{x\xi} G(0, x) = 0, \quad G_{,\xi}(1, x) = 0. \quad (3.17)$$

Then, for any order

$$u_x^{(k)} = G_{\xi x} * f_\xi^{(k)}. \quad (3.18)$$

Notice that the second derivative of G, known as the "Modified Green function" (Fokin, 1996; Kröner, 1986) is a Dirac operator itself, and cannot be ignored.

3.2. Linearly varying modulus

Consider the simple case

$$E_x = 1 + x \quad (3.19)$$

For which an exact solution exists

$$u(x) = p \ln(1 + x). \quad (3.20)$$

The FPM solution for the first three steps (i.e., up to the second-order) is

$$u_{x(\mathrm{FPM})} = \sum_{k=0}^{2} u^{(k)} = \frac{px}{2} - \frac{px}{6}(x - 2) + \frac{px}{54}(4x^2 - 9x + 6) = \frac{px}{3}\left(\frac{17}{6} - x + \frac{2x^2}{9}\right). \quad (3.21)$$

The three terms (3.21b) above correspond to $u^{(0)}$, $u^{(1)}$ and $u^{(2)}$, respectively. A comparison between the exact (3.19) and the FPM (3.20) solution is shown in Fig. 1. The accuracy and convergence is clearly seen, from the zero-order straight line solution of a homogeneous case to the second-order results with approximately 1% error.

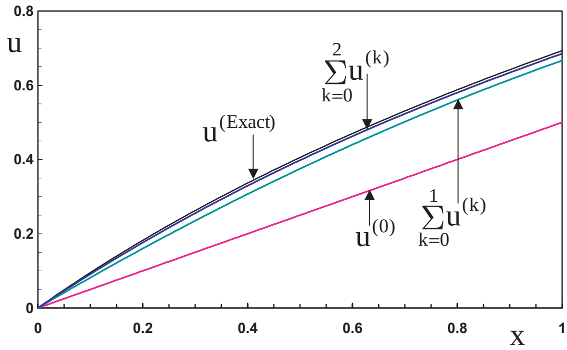

Fig. 1. Zero-, first- and second-order results (3.21) of the FPM for a non-uniform rod (3.19), compared with the exact displacement field (3.20).

3.3. Nonlinear modulus in space

To further examine the capabilities of the FPM, we consider next a nonlinear example for the function E

$$E_x = 1 + x^2. \tag{3.22}$$

The exact solution is

$$u_x = p \cdot \mathrm{arctg}(x). \tag{3.23}$$

Following the above procedures, the three step FPM solution is

$$u_{x(\mathrm{FPM})} = \sum_{k=0}^{2} u^{(k)} = \frac{px}{2} - \frac{px}{8}(x^2 - 3) + \frac{px}{160}(9x^4 - 20x^2 + 15) = \frac{px}{4}\left(\frac{31}{8} - x^2 + \frac{9x^4}{40}\right). \tag{3.24}$$

The FPM relative error, for the first three accuracy orders is shown in Fig. 2. It is seen that three terms solution have an error of less than 3%, which is more than for the previous case due to the nonlinearity of $E(x)$.

3.4. Discontinuous field

The purpose of this section is to examine the FPM for cases when material properties (here modulus E) have a "jump" (discontinuity), which is common in composite materials. Consider a modulus having a step function

$$E(x) = \begin{cases} E_1, & 0 < x < s, \\ E_2, & s < x < 1. \end{cases} \tag{3.25}$$

Boundary conditions are of (3.2). The exact solution is

$$u(x) = p \cdot \begin{cases} xE_1^{-1}, & 0 < x < s, \\ xE_2^{-1} + s(E_1^{-1} - E_2^{-1}), & s < x < 1. \end{cases} \tag{3.26}$$

The FPM solution of Eq. (3.1) is

$$u^{(k)}(x) = p \cdot \beta^{(k)} \begin{cases} x, & 0 < x < s, \\ s, & s < x < 1, \end{cases} \tag{3.27}$$

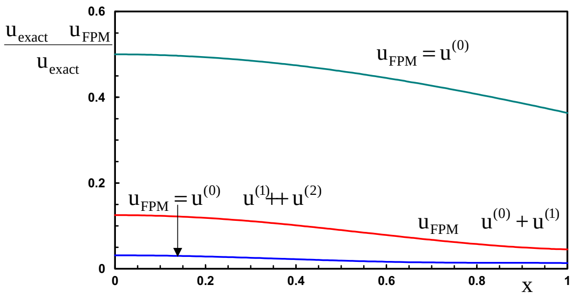

Fig. 2. Relative errors of the displacement field for zero-, first- and second-order results (3.24) of the FPM for a non-uniform rod (3.22).

where

$$\beta^{(k)} = \frac{(1-\alpha)^k (1-s)^{k-1}}{E_2(1-s+\alpha s)^k}, \quad \alpha = \frac{E_1}{E_2}. \tag{3.28}$$

Therefore

$$u(x) = \frac{p}{E_2}\begin{cases} x\left(1+\sum_{k=1}^{\infty}\beta^{(k)}\right), & 0 < x < s, \\ \left(x + s\sum_{k=1}^{\infty}\beta^{(k)}\right), & s < x < 1. \end{cases} \tag{3.29}$$

A comparison between the exact and the FPM solutions ($s = 0.5$) for the zero-, first- and second-order accuracy is shown in Fig. 3. It is seen that the solutions for all orders obey the boundary conditions (zero displacements at $x = 0$ and the exact slope for $x > 0.5$). It is important to emphasize, that the FPM solution is obtained by solving *only one* governing equation, while the exact solution (3.26) is a consequence of solving two separate regions simultaneously.

4. Case study 2: elastic beams with variable stiffness

Examine a higher (fourth) order problem of a beam with non-uniform stiffness. The Euler–Bernoulli beam equation is

$$L(K,w) = (K(x)\cdot w_{,xx})_{,xx} = q(x), \tag{4.1}$$

$w(x)$ is the beam deflection. $K(x) = E(x)I(x)$ is the bending stiffness, where $E(x)$ is Young's modulus and $I(x)$ the moment of inertia of the cross-section.

Consider the case of a uniform loading ($q(x) = q$), one end of the beam is rigidly supported and the other end is simply supported. The boundary conditions are:

$$w(0) = w_{,x}(0) = w(1) = w_{,xx}(1) = 0. \tag{4.2}$$

Choosing

$$\langle K\rangle = K * 1, \quad K = \langle K\rangle + K'. \tag{4.3}$$

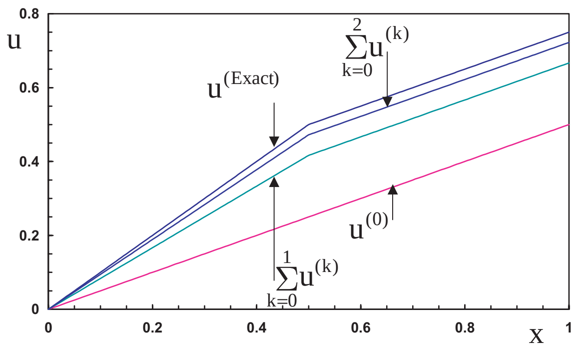

Fig. 3. Zero-, first- and second-order results of the FPM for a discontinuous (step function at $x = 0.5$) modulus of a rod in tension (3.25), compared with the exact displacement field (3.26).

The zero-order equation is

$$L(\langle K\rangle, w^{(0)}) = \langle K\rangle w^{(0)}_{,xxxx} = q. \tag{4.4}$$

where the b.c's for $w^{(0)}$ are as in (4.2). Then, the recursive differential equations are

$$\begin{aligned} &\langle K\rangle w^{(k)}_{,xxxx} + (K'_x w^{(k-1)}_{,xx})_{,xx} = 0, \\ &w^{(k)}(0) = w^{(k)}_{,x}(0) = w^{(k)}(1) = w^{(k)}_{,xx}(1) = 0. \end{aligned} \tag{4.5}$$

The solution of (4.5) is

$$w^{(k)}(\xi) = G_{x\xi,xx} \mathop{*}_{x=0}^{1} K'_x w^{(k-1)}_{,xx}, \tag{4.6}$$

where the Green function is

$$G(x,\xi) \equiv G_{x\xi} = \frac{1}{12\langle K\rangle}\begin{cases} -x^2(\xi-1)[\xi(\xi-2)(x-3)-2x], & 0<x<\xi, \\ -\xi^2(x-1)[\xi(x(x-2)-2)-3x(x-2)], & \xi<x<1. \end{cases} \tag{4.7}$$

To evaluate the FPM accuracy, consider the nonlinear example

$$K = 1 + x^2. \tag{4.8}$$

By direct integration, the exact analytical solution is

$$w_x = \frac{q}{8(\ln 2 - 1)}(R_x - 2P_x \cdot \mathrm{arctg}(x) + Q_x \cdot \ln(1+x^2)), \tag{4.9a}$$

where P, Q and R are polynomials of x

$$\begin{aligned} R_x &= 2x((1-\pi+16\ln 2) + (\ln 2 - 1)x), \\ P_x &= (1-\pi+2\ln 2) + (\pi - 5 + 2\ln 2)x, \\ Q_x &= (\pi - 5 + 2\ln 2) + (1-\pi+2\ln 2)x. \end{aligned} \tag{4.9b}$$

The three terms FPM approximation solution ($n = 2$) is

$$w_{(n=2)} = \sum_{k=0}^{2} w^{(k)} = \frac{3q \cdot x^2(x-1)}{716800}(900x^5 - 600x^4 - 3540x^3 + 2697x^2 + 10502x - 13647). \tag{4.10}$$

Eq. (4.10) is compared to the exact solution (4.9) in Fig. 4. It is seen that the second-order ($n = 2$) FPM solution fully corresponds with the exact solution. Considering the irregular exact solution of (4.9), the FPM accuracy is striking.

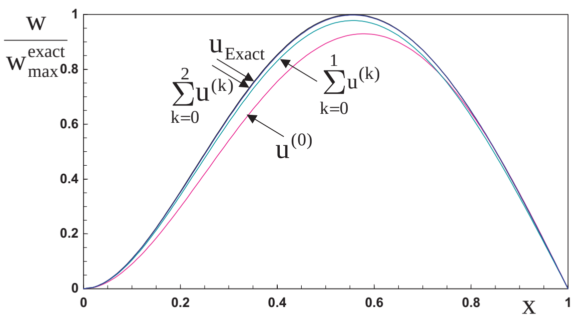

Fig. 4. Zero-, first- and second-order results of the FPM deflections (4.10) for a non-uniform beam (4.8), compared with the exact deflections (4.9).

4.1. Comparison with other approximation methods (Rayleigh–Ritz and Galerkin)

In comparing the FPM accuracy with existing approximation methods, it is important to use similar shape functions and identical number of elements in the series. Since (4.10) is a polynomial of degree 8, the Rayleigh–Ritz (RR) solution is taken as a polynomial of the same degree (or higher). Therefore, we choose for the RR solution the form

$$w^{\mathrm{RR}}(x) = x^2(x-1)\cdot\sum_{i=0}^{7} a_i x^i. \tag{4.11}$$

Solution (4.11) fulfills the three geometrical boundary conditions in (4.2) automatically. The coefficients a_i are found by applying the principle of minimum potential energy

$$\pi = K * w_{,xx}^2 - q * w, \quad \pi_{,a_i} = 0. \tag{4.12}$$

In the Galerkin method, we choose an orthonormal polynomial which satisfies all four conditions of (4.2)

$$w^{\mathrm{G}}(x) = \sum_{i=0}^{5} w_i^{\mathrm{G}}(x), \quad w_k^{\mathrm{G}} = x^2(x-1)^3\cdot\sum_{i=0}^{k} b_i^{(k)} x^i, \tag{4.13a}$$

$$w_k^{\mathrm{G}} * w_m^{\mathrm{G}}|_{k\neq m} = 0, \quad \|w_k^{\mathrm{G}}\|_2 = 1. \tag{4.13b}$$

The unknown coefficients are found by substituting w^{G} in the differential equation and convoluting separately with each term

$$(L(w^{\mathrm{G}}) - q) * w_k^{\mathrm{G}} = 0. \tag{4.14}$$

For design (strength) purposes, it is fruitful to compare the accuracy of the above methods by calculating the bending moment distribution ($M = Kw_{,xx}$), which is proportional to the maximum stresses along the beam. Results are shown in Fig. 5 and compared to the exact solution. It is seen that the FPM and RR accuracy is excellent, while the Galerkin method is poor. The reason for the RR quality is the proper choice of base functions; for example, a poor accuracy was obtained for trigonometric series

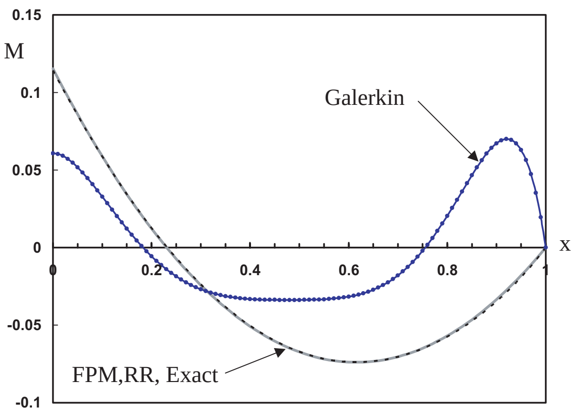

Fig. 5. Second-order solution of the FPM bending moment field $M = EIw_{,xx}$, for a non-uniform beam (4.8), compared with (a) exact solution; (b) approximate Rayleigh–Ritz solution (4.11) and (c) approximate Galerkin solution (4.13).

even for higher number of shape functions (not shown here). The poor accuracy of the Galerkin method even for the same shape functions is due to the different convergence criteria, which is built in the method.

5. Case study 3: deflection of a beam on elastic foundation

The main objective of Sections 3 and 4 was to demonstrate the accuracy of the FPM. Therefore, problems for which exact solution exists were chosen. In this section a more complex problem is analyzed, for which there is no closed from solution. Thus, the accuracy of the FPM is compared to finite differences solution. Problems of this type have been studied by other methods (Frantziskonis and Breysse, 2003) The governing equation for an Euler-Bernoulli beam on an elastic foundation is

$$(K \cdot w_{,xx})_{,xx} + k \cdot w = q, \tag{5.1}$$

where K, k, q and w are the beam bending stiffness, the elastic foundation stiffness, external loading and beam deflection, respectively. In the following, the deflection of a semi-infinite beam on elastic foundation is studied for two cases. Case I: a beam with non-homogeneous bending stiffness of the form

$$K(x) = \begin{cases} 1 + 3x^2, & 0 < x < 1, \\ 4, & x \geqslant 1, \end{cases} \quad k(x) = 4. \tag{5.2}$$

Case II: an elastic foundation with non-homogeneous stiffness of the form

$$K(x) = 4, \quad k(x) = \begin{cases} 1 + 3x^2, & 0 < x < 1, \\ 4, & x \geqslant 1. \end{cases} \tag{5.3}$$

Both cases were solved for

$$q(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & x \geqslant 1 \end{cases} \tag{5.4}$$

and the boundary conditions

$$w(0) = w_{,xx}(0) = w(\infty) = w_{,x}(\infty) = 0. \tag{5.5}$$

The purpose of choosing a complete similarity between the heterogeneity of the bending stiffness in the first case and the elastic foundation stiffness of the second case is to compare the FPM accuracy between the two non-uniformities. The three terms (second-order) FPM solution is of the form

$$w^{(n=3)} = w^{(0)} + w^{(1)} + w^{(2)} \tag{5.6}$$

where

$$w^{(0)} = \begin{cases} \dfrac{\mathrm{e}^{-p(1+x)}[\mathrm{e}^{2px}\cos(p(1-x)) + \cos(p(1+x)) - 2\mathrm{e}^{p(1+x)}]}{8\langle K\rangle p^4} & \text{for } x < 1, \\ -\dfrac{\mathrm{e}^{-}px[\cosh(p)\sin(p)\sin(px) + \sinh(p)\cos(p)\cos(px)]}{4\langle K\rangle p^4} & \text{for } x \geqslant 1, \end{cases} \tag{5.7}$$

$$\langle K\rangle = 4, \quad \langle k\rangle = 4, \quad p \equiv \left(\frac{\langle k\rangle}{4\langle K\rangle}\right)^{1/4} = 2^{-1/2}. \tag{5.8}$$

The FPM expressions for $w^{(1)}$ and $w^{(2)}$ was obtained analytically (details are not given here). Since the explicit solution is too long, only the dependent functions of these expressions are given below

$$w^{(1)} = w^{(1)}\begin{pmatrix} \cos(p(1-x)), \cos(p(1+x)), \cos(p(3-x)), \cos(p(3+x)) \\ \sin(p(1-x)), \sin(p(1+x)), \sin(p(3-x)), \sin(p(3+x)) \\ \exp(5xp), \exp(6xp), \exp(7xp) \\ x, x^2, x^3 \end{pmatrix}, \tag{5.9}$$

$$w^{(2)} = w^{(2)}\begin{pmatrix} \cos(p(1-x)), \cos(p(1+x)), \cos(p(3-x)), \cos(p(3+x)) \\ \sin(p(1-x)), \sin(p(1+x)), \sin(p(3-x)), \sin(p(3+x)) \\ \cos(p(5-x)), \cos(p(5+x)), \sin(p(5-x)), \sin(p(5+x)) \\ \exp(5xp), \exp(6xp), \exp(7xp) \\ x, x^2, x^3, x^4, x^5 \end{pmatrix} \tag{5.10}$$

A comparison of the FPM solutions for the first three orders with the Finite Difference results is shown in Fig. 6. The error is less than 10% and 3% for cases I and II, respectively. The convergence characteristics of the two cases are different—while in case II the error is always positive in x, in case II it is not. Also, the maximal error is at the maximum deflection point $x = 0$ for case II, but not for case I.

6. Case study 4: heat conduction

The heat conduction problem of a one dimensional rod is considered next. The governing equation for the temperature field $u(x,t)$ is

$$u_{xt,t} = (a_x u_{xt,x})_{,x} = a_x u_{xt,xx} + a_{x,x} u_{xt,x}, \tag{6.1}$$

$a(x)$ is a positive, non-uniform material property, taken here as

$$a = \frac{k}{\gamma\rho} = 1 + x^2. \tag{6.2}$$

k is the coefficient of internal conduction, γ is the specific heat of the body and ρ is the density. Boundary conditions are of a constant temperature type

$$u(x = 0, t) = T_0, \quad u(x = 1, t) = T_1. \tag{6.3}$$

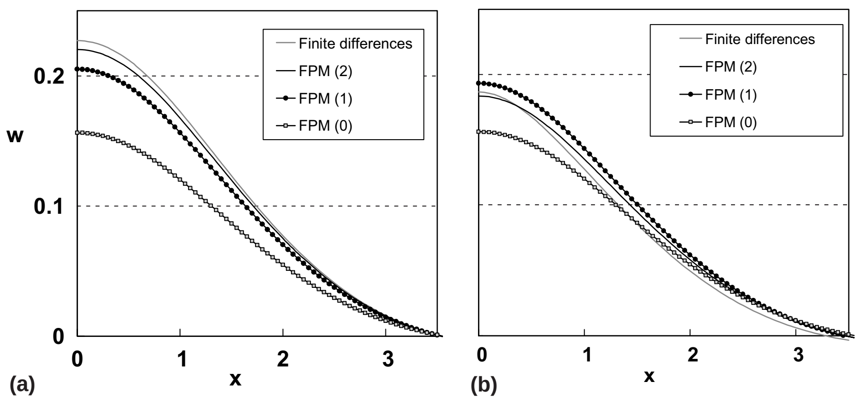

Fig. 6. FPM solution for the deflection of a beam on elastic foundation: (a) non-homogeneous elastic foundation (5.3) and (b) non-homogeneous beam stiffness (5.2).

This problem is essentially 2D (space and time), and there are many realizations by which the FPM can be tested. For convenience, the initial condition (temperature distribution $u(t=0,x)$) is taken to be *identical* to the exact asymptotic stationary field:

$$u(x,t=0)=\frac{\pi}{4}(T_1-T_0)\cdot\text{arctg}(x)+T_0. \tag{6.4}$$

Therefore, (6.4) is also the exact temperature field for any time t, i.e.

$$u(x,t)=u(x,t=0). \tag{6.5}$$

Thus, although the exact field is not a function of time, it is expected that the finite FPM series will be time dependent, since the zero-order approximation "starts" from the homogeneous case. This "artificial" time dependency should decay as more terms are added to the series. Our aim is to examine how fast the FPM converges to the asymptotic solution. The zero-order FPM equation of (6.1) is

$$u^{(0)}_{,t}=\langle a\rangle u^{(0)}_{,xx}, \tag{6.6}$$

$\langle a\rangle$ is found by averaging

$$\langle a\rangle=a\underset{x=0}{\overset{1}{*}}1=\frac{4}{3}. \tag{6.7}$$

The initial and boundary conditions have the same form as (6.3) and (6.4). This problem is solved by Fourier's method, which yields (see Appendix A)

$$u^{(0)}(x,t)=T_0+(T_1-T_0)x+\sum_{m=1}^{\infty}A_m\mathrm{e}^{-m^2\cdot t/\tau}\sin\pi mx,\quad \tau=\frac{1}{\pi^2\langle a\rangle}, \tag{6.8}$$

where A_m is given by (A.12). Following the FPM procedures above, the recursive differential equation is

$$u^{(k)}_{,t}=\langle a\rangle u^{(k)}_{,xx}+\left(a'_x u^{(k-1)}_{,x}\right)_{,x}, \tag{6.9}$$

where

$$a'_x=a-\langle a\rangle=x^2-1/3. \tag{6.10}$$

The initial and boundary conditions of (6.9) are homogeneous

$$u^{(k)}(x,0)=0,\quad u^{(k)}(0,t)=u^{(k)}(1,t)=0. \tag{6.11}$$

The solution of equation (6.9) is solved formally by assuming that $u^{(k)}(x,t)$ can be expressed as a Fourier's series of the form

$$u^{(k)}(x,t)=\sum_{m=1}^{\infty}B^{(k)}_m(t)\cdot X_m(x),\quad X_m=\sin m\pi x. \tag{6.12}$$

The boundary conditions (6.11) are satisfied automatically. Substituting (6.12) in (6.1)

$$\sum_{m=1}^{\infty}\left(\dot B^{(k)}_m X_m-\langle a\rangle B^{(k)}_m X_{m,xx}\right)=\sum_{m=1}^{\infty}\left(\dot B^{(k)}_m+\lambda_m^2\langle a\rangle B^{(k)}_m\right)X_m=a'\sum_{m=1}^{\infty}B^{(k-1)}_m X_{m,x} \tag{6.13}$$

Multiplying by X_n and integrating over $(0,1)$, we obtain

$$\dot B^{(k)}_n+\lambda_n^2\langle a\rangle B^{(k)}_n=f^{(k)}(t). \tag{6.14}$$

$$f^{(k)}(t)=\sum_{m=1}^{\infty}B^{(k-1)}_m\int_0^1 a'_{,x}X_{m,x}X_n\,\mathrm{d}x. \tag{6.15}$$

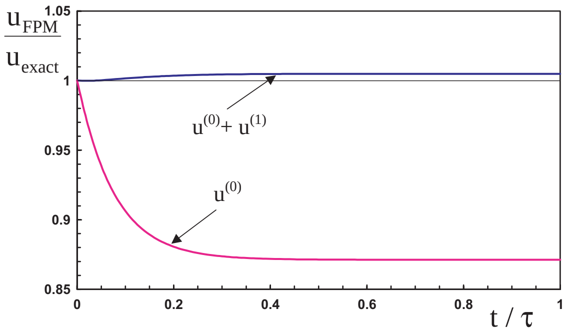

Fig. 7. Temperature vs time at $x = 0.5$ for the zero- and first-order FPM solutions, as compared to the exact solution.

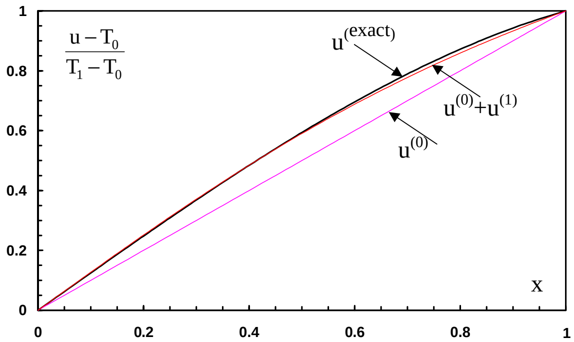

Fig. 8. Temperature field after large time ($t = 1$) for the zero- and first-order FPM solutions, as compared to the exact solution.

In order to satisfy the initial conditions in (6.11) we insert $B_n^k(0) = 0$ and obtain from (6.14) and (6.15)

$$B_n^{(k)}(t) = \mathrm{e}^{-n^2 \cdot t/\tau} \int_0^t \mathrm{e}^{n^2 \cdot t'/\tau} f^{(k)}(t')\,\mathrm{d}t' \tag{6.16}$$

We know the solution for $k = 0$, from which $B_n^0(t)$ is found

$$B_m^0(t) = c_m + A_m \mathrm{e}^{-m^2 \cdot t/\tau}, \quad c_m = \int_0^1 (T_0 + (T_1 - T_0)x) \cdot X_m(x)\,\mathrm{d}x. \tag{6.17}$$

Using (6.17) the recursion relation (6.15) can be calculated, from which $u^{(k)}(x,t)$ for any k is found.

Fig. 7 shows the temperature field $u(x = 0.5,t)$ for the first ($u^{(0)}$) and second ($u^{(0)} + u^{(1)}$) order FPM results, normalized by the exact solution (6.4), which was calculated by the first 10 terms of the Fourier series. The two terms result has an error of about 1%. In Fig. 8, the temperature distribution $u(x, t = 1)$ is shown for the same zero- and first-order FPM results. Notice the straight line for the homogeneous case, which is the expected exact solution for large times. The accuracy of the FPM for the two terms approximation is clear.

7. Discussion and conclusions

The FPM is found to be a powerful tool for analytical analysis of linear and non-homogeneous materials. The accuracy is satisfactory even for "not so small" variations in material properties. For example, one percent accuracy was obtained with a second-order solution for a modulus non-uniformity of 100% (relative values ranging from 1 to 2). Moreover, the FPM is generally more accurate than the common Galerkin (G) and Rayleigh–Ritz (RR) methods. Comparing with (G&RR) yields the following conclusions:

(a) The accuracy of the G&RR methods depends on the shape functions chosen. This is in contrast to the FPM, which produces "natural" functions that are generic to each problem.
(b) The RR method always converges (the accuracy is always improved by increasing the number of terms). On the other hand, the convergence of the FPM is not guaranteed for extreme cases of heterogeneity (holes).
(c) The G&RR methods are implicit. Thus, when large number of terms is used, the solution exhibits inherent numerical errors. This is not expected for the FPM, which yields a closed form explicit solution.
(d) The FPM needs as a basis the solution for the homogeneous case. It is expected that if this basis is not exact, the FPM could be still used to find approximate solutions for the non-homogeneous problem. In these cases the (heterogeneous) solution accuracy will follow the accuracy of the homogeneous solution in hand.

The major differences between the FPM and regular (small parameter) perturbation methods (RPM) are

(a) The RPM assumes that the solution $u(x)$ takes the form

$$u = u^{(0)} + u^{(1)} + u^{(2)} + \cdots \tag{7.1}$$

The terms in the series are of increasing order with respect to a "very small" parameter ε. The FPM, on the other hand, is not constrained to small perturbations in the same sense. For example, solving the problem of Section 5 using RPM is impossible. Moreover, it is frequently very difficult (even impossible) to define this "very small" parameter a priori.
(b) In the FPM process, the functional derivatives of the unknown function u, with respect to E (the non-uniform material property), depend on the solution of the homogeneous problem and are independent of the specific morphology of the heterogeneous case. Therefore, the solution of u for each problem is found by merely convoluting the functional derivatives by the n-point morphology functions of E (see (2.8)). In this sense the functional derivatives of u with respect to E can be considered as "Green functions of heterogeneity" at different orders, which "operate" on the morphology at hand. This characteristic, not only contributes to the physical insight, but also allows us to solve the problem *only once* for *any* heterogeneity.
(c) The FPM can be easily applied to structures with stochastic material properties too (Altus, 2001; Altus and Givli, 2003; Altus and Totry, 2003).
(d) The accuracy of the FPM is affected by the "reference" homogeneous property near which the solution is expanded. This effect is demonstrated in Fig. 9, where the relative error of the FPM solution for study case 3.3 is shown as a function of the reference homogeneous property E_0 for the first four orders of the solution. It is seen that the accuracy is significantly affected by the choice of E_0, especially for lower orders of the functional perturbation series. Interestingly, it was found that for this example, the optimal value of E_0 approaches $\langle E \rangle$as the order of the FPM increases. Finding the "best" reference property is an important subject of optimization which is currently under investigation.

The RHS ($f^{(i)}$) functions which appear in each order (see (2.16) and (2.22)) can be interpreted as pseudo-body forces, which "correct" the inconsistencies of all previous approximations. This is a reminder that a non-homogeneous problem with one set of external loading can be transformed into a homogeneous problem loaded by an associated field of "eigenstresses".

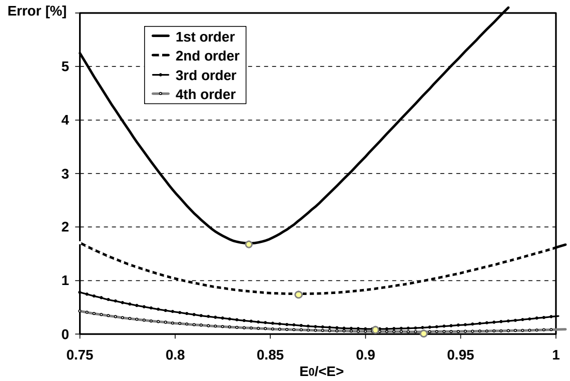

Fig. 9. The relative error of the FPM solution for study case 3.3 as a function of the reference homogeneous property E_0. Minimum points are emphasized by a circle. Note the convergence of the optimum to the $\langle E\rangle$ point.

Finally, the capability of solving problems with non-continuous material properties is especially encouraging for future research, since many composites and granular materials exhibit moduli fields of this type.

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Appendix A

The problem to be solved is

$$u^{(0)}_{,t} = \langle a\rangle u^{(0)}_{,xx}, \tag{A.1}$$

where

$$\langle a\rangle = \int_0^1 (1+x^2)\,\mathrm{d}x = \frac{4}{3}. \tag{A.2}$$

Initial and boundary conditions are

$$\begin{aligned} u^{(0)}(x,t)|_{t=0} &= \frac{(T_1 - T_0)\pi}{4}\operatorname{arctg}x + T_0, \\ u^{(0)}(x,t)|_{x=0} &= T_0, \quad u^{(0)}(x,t)|_{x=1} = T_1. \end{aligned} \tag{A.3}$$

Using the Fourier method, define $v(x,t)$ by

$$u^{(0)}(x,t) = z(x) + v(x,t). \tag{A.4}$$

$z(x)$ is the linear field of the steady-state equation

$$z(x) = (T_1 - T_0)x + T_0. \tag{A.5}$$

$v(x,t)$ is the solution of

$$\begin{aligned} &v_{,t} = \langle a\rangle v_{,xx}, \\ &v(0,t) = v(1,t) = 0, \quad v(x,0) = \frac{T_1 - T_0}{\pi}(4\operatorname{arctg}x - \pi x). \end{aligned} \tag{A.6}$$

A particular solution of (A.6) is assumed in the form

$$T(t)\cdot X(x). \tag{A.7}$$

Inserting in (A.6) yields

$$\begin{aligned} &\dot{T} + \lambda^2\langle a\rangle T = 0, \\ &X'' + \lambda^2 X = 0, \quad X(0) = X(1) = 0. \end{aligned} \tag{A.8}$$

Therefore

$$\lambda_k = \pi k, \quad X_k(x) = \sin\lambda_k x \tag{A.9}$$

and

$$v(x,t) = \sum_{k=1}^{\infty} A_k \mathrm{e}^{-k^2\cdot t/\tau}\sin\pi kx = \sum_{k=1}^{\infty} A_k T_k X_k. \tag{A.10}$$

The coefficients A_k are determined from initial conditions:

$$v(x,0) = \sum_{k=1}^{\infty} A_k X_k(x) = \frac{T_1 - T_0}{\pi}(4\operatorname{arctg}x - \pi x), \tag{A.11}$$

which, by the Fourier series, implies

$$A_m = \frac{2(T_1 - T_0)}{\pi}\int_0^1 (4\operatorname{arctg}x - \pi x)\cdot X_m(x)\,\mathrm{d}x. \tag{A.12}$$

Finally, the solution of (A.1)–(A.3) is obtained in a series form as:

$$u^{(0)}(x,t) = T_0 + (T_1 - T_0)x + \sum_{m=1}^{\infty} A_m \mathrm{e}^{-m^2\cdot t/\tau}\sin\pi mx. \tag{A.13}$$

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